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Journal of Sound and Vibration 304 (2007) 530–540
JOURNAL OF SOUND AND VIBRATION
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An asymptotic solution to transverse free vibrations of variable-section beams

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Received 6 June 2006; received in revised form 16 December 2006; accepted 28 February 2007
Available online 24 April 2007

Abstract

The transverse free vibration of a class of variable-cross-section beams is investigated using the Wentzel, Kramers, Brillouin (WKB) approximation. Here the governing equation of motion of the Euler–Bernoulli beam including axial force distribution is utilized to obtain a singular differential equation in terms of the natural frequency of vibration and a WKB expansion series is applied to find the solution. Based on this formulation, a closed form solution is obtained for determination of natural vibration mode shapes and the corresponding frequencies. The first four terms of this asymptotic solution are simplified for homogenous beams to give a compact third-order WKB approximation. Next, the resulting solution is employed to determine the natural frequencies and mode shapes of some examples with and without axial force distribution. The results are then been compared with those in the literature and very good agreement is achieved.

1. Introduction

Engineering problems are often described by partial differential equations and in most cases it is extremely difficult to find their closed form solutions. Consequently, more efforts have been mainly concentrated on approximate numerical methods such as finite element, finite difference and boundary element methods which are widely used to solve such types of problems. Nevertheless, besides all advantages of using such numerical methods, closed form solutions appear more appealing because of their account of the physics of the problems and convenience for parametric studies. Furthermore, closed form solution may be utilized for developing more efficient and accurate numerical methods.

The other advantage of closed form solutions is their significance in the field of inverse problems. A close form solution can be more useful than the numerical methods to determine or design the characteristic of an engineering system (e.g. geometry), to achieve a prescribed behavior and function. Designing a structure for desired mode shapes of free vibration, in addition to natural frequencies, for engineering applications (e.g. resonant-mode micro sensors and actuators, acoustics, manufacturing tools, etc.), are some examples of this issue [1].

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2007.02.030

Investigation of vibrations of beam-like structures, as fundamental structural elements in many engineering applications, has been an interesting field of study for many researchers. Due to the wide range of applications and the specific geometric feature of beams, in which one dimension is much larger than the other two, various beam models have been employed to simulate the structural dynamics of aircraft wings, helicopter blades, spacecraft antennas, robot arms, towers and for many other industrial applications. Numerous methods such as experimental, analytical and numerical methods have been developed and used to analyze the structural dynamics of beam-like structures. In this respect, the modal analysis is a well-known practical technique for investigation of the dynamic response and vibrations of beams. The modal approach gives the solution in a series in terms of natural mode shapes and the corresponding generalized coordinates. Subsequently, one needs first to determine the natural mode shapes and frequencies of free vibrations, analytically or numerically for using such techniques.

A lot of research, much of it by Prescott [2] and Meirovitch [3], has been done on the free vibrations of beams. Indeed transverse free vibrations of non-uniform beams have been studied by numerous researchers in both aeronautical and mechanical engineering fields either analytically or numerically. Added to this, several analytical solutions, most of which are applied for linearly tapered beams, have been represented in terms of orthogonal polynomials [4], Bessel functions [5–7], hyper geometric series [8], power series by Frobenius method [9,10], Differential transform method (DTM) [11] and classical Jacobi polynomials. Some analytical solutions have also been developed for several classes of non-uniform beams, based on uniform beam theory [12]. The dynamic stiffness method has been utilized to obtain natural frequencies and mode shapes of rotating tapered beam in Refs. [13,14] in which several cases are studied with various rotational frequencies and hub diameters.

On the other hand, a wide range of approximate and numerical solutions such as Rayleigh-Ritz, Gallerkin, finite difference, finite element and spectral finite element methods have been used to obtain the natural vibration characteristics of variable-section beams [15–20].

The idea of the Wentzel, Kramers, Brillouin (WKB) approximation for solving differential equations has been revitalized during the 1920s, motivated by the need of having explicit approximate analytical representations of solutions to the 1D stationary Schrödinger equation in quantum mechanics [21] and recently has been developed for matrix differential equation and semi-discretized partial differential equations [22] and is used in many engineering fields. In the field of structural dynamics, the WKB method has been utilized in combination with the dynamic stiffness method to investigate the free vibration of marine risers [23]. In this reference a second-order WKB approximation is used to develop a WKB-based dynamic element stiffness matrix for free vibration evaluations. Complex WKB method has been also used for studying some structural dynamics problems in thin cylindrical shells [24,25].

This paper generally deals with the transverse free vibrations of typical truncated non-uniform Euler–Bernoulli beams by using a WKB global approximation. The asymptotic solution is obtained for a general Euler–Bernoulli beam including the axial force and simplified for the case of a homogenous beam. The subsequent solution is used to investigate two illustrative examples for determination of natural mode shapes and frequencies.

2. Governing equation

The governing equation for investigation of free transverse vibrations of a non-uniform Euler–Bernoulli beam of length L is a fourth-order linear differential equation with variable coefficients which can be expressed as follows [3]:

$$\left(\alpha^4 w''(x,t)\right)'' - (pw'(x,t))' + \beta^4 \ddot{w}(x,t) = 0, \quad 0<x<L, \tag{1}$$

where $w(x,t)$ is the transverse displacement and p is the axial force distribution over the beam which is positive in tension. The operators $(\ )'$ and $(\dot{\ })$ denote the partial derivatives with respect to x and t, respectively and

$$\alpha = \sqrt[4]{EI(x)}, \quad \beta = \sqrt[4]{m(x)}, \tag{2}$$

where $EI(x)$ and $m(x)$ are the bending stiffness and mass distribution of the beam, respectively. Assuming harmonic oscillations yields

$$w = \bar{w}(x)\exp(\lambda t), \quad \lambda = \mathrm{i}\omega, \ \ \mathrm{i} = \sqrt{-1}, \tag{3}$$

where $\bar{w}(x)$ is the amplitude of motion and ω the angular frequency.

Substituting Eq. (3) into Eq. (1) yields the following boundary value problem:

$$\left(\alpha^4\bar{w}''(x)\right)'' - (p\bar{w}'(x))' - \omega^2\beta^4\bar{w}(x) = 0, \tag{4}$$

which its boundary conditions are determined in consistency with the physical conditions at the two ends of the beam and are chosen as [3]

$$\left(\alpha^4\bar{w}''\right)' = 0 \quad \text{or} \quad \bar{w} = 0, \quad \text{at } x = 0, L \tag{5}$$

and either

$$\alpha^4\bar{w}'' = 0 \quad \text{or} \quad \bar{w}' = 0, \quad \text{at } x = 0, L. \tag{6}$$

3. Solution approach

3.1. Derivation of WKB approximation

In order to use the WKB approximation, the first step is to transfer Eq. (4) into a singular perturbation differential equation. For this end, the governing equation of motion is expressed as

$$\varepsilon^4\left(\alpha^4\bar{w}''(x)\right)'' - \varepsilon^4(p\bar{w}'(x))' - \beta^4\bar{w}(x) = 0, \tag{7}$$

in which

$$\varepsilon^2 = \omega^{-1}. \tag{8}$$

In most of engineering applications, the natural frequencies of the beams are greater than unity and the parameter ε has a small value. As the result, for a truncated beam, one can deal with Eq. (7) as a singular perturbation boundary value problem with a dispersive nature. Thus, the WKB method may be utilized to derive an asymptotic expression for the solution of Eq. (7) as a function of the perturbation parameter ε. This expression is written in the following exponential form [26,27]:

$$\bar{w}(x) \simeq \exp\left(\frac{1}{\delta}\sum_{k=0}^{n} S_k(x)\delta^k\right), \quad \delta \to 0, \tag{9}$$

where the conditions that must be satisfied for Eq. (9) to be applicable are

$$\begin{aligned} &\delta S_{k+1}(x) \ll S_k(x), \quad \delta \to 0, \\ &\delta^k S_{k+1}(x) \ll 1, \qquad \delta \to 0. \end{aligned} \tag{10}$$

Setting δ proportional to ε, substituting Eq. (9) into Eq. (7) and dividing by the exponential factors, yields an ordinary differential equation in terms of the S_k functions and their derivatives and power terms of ε. By dominate balance of terms with the same order of magnitude, one can obtain a sequence of identical equations which determine S_k functions as follows:

$$\varepsilon^{-1} : S_0'^4 = \frac{\beta^4}{\alpha^4}, \tag{11}$$

$$\varepsilon^0 : S_1' = -\frac{3S_0'}{2S_0'} - 2\frac{\alpha'}{\alpha}, \tag{12}$$

$$\begin{aligned} \varepsilon^1 : S_2' = &-\frac{3S_0''^2}{4S_0'^3} - 6\frac{\alpha' S_0''}{\alpha S_0'^2} - \frac{S_0'''}{S_0'^2} - 3\frac{S_0'' S_1'}{S_0'^2} - \frac{3S_1'^2}{2S_0'} - \frac{\alpha''}{\alpha S_0'} - 6\frac{\alpha' S_1'}{\alpha S_0'} \\ &- \frac{3S_0''}{2S_0'} - 3\frac{\alpha'^2}{\alpha^2 S_0'} + \frac{p}{4\alpha^4 S_0'}, \end{aligned} \tag{13}$$

$$\varepsilon^2 : S_4' = \cdots. \tag{14}$$

Eq. (11) is a first-order nonlinear differential equation and can be simply solved. The other equations are linear and determine the higher order terms in the expansion. Since Eq. (11) has four solutions, there are four fundamental solutions for $\bar{w}(x)$ and the general solution of Eq. (7) is a linear combination of the fundamental set of solutions. This general solution can be simplified as

$$\bar{w}(x) = \mathrm{e}^{A_1}(c_1 \sinh A_2 + c_2 \cosh A_2) + \mathrm{e}^{A_3}(c_3 \sin A_4 + c_4 \cos A_4), \tag{15}$$

in which

$$\begin{aligned} A_1 &= \sum_{k=0}^{n} \varepsilon^{2k} \phi_{(2k+1)}, \\ A_2 &= \sum_{k=0}^{n} \varepsilon^{2k-1} \phi_{(2k)}, \\ A_3 &= \sum_{k=0}^{n} (-1)^k \varepsilon^{2k} \phi_{(2k+1)}, \\ A_4 &= \sum_{k=0}^{n} (-1)^k \varepsilon^{2k-1} \phi_{(2k)}, \end{aligned} \tag{16}$$

where

$$\phi_i = S_i \tag{17}$$

while S_0 is set to be equal to

$$S_0 = \int_0^x \frac{\beta}{\alpha}\, \mathrm{d}\xi. \tag{18}$$

Inserting the obtained solution (Eq. (15)) into the boundary condition equations yields the following set of linear equations:

$$\mathbf{Ac} = 0, \quad \mathbf{c} = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}^{\mathrm{T}}. \tag{19}$$

This homogeneous linear system has non-trivial solutions if and only if the determinant of the corresponding coefficient matrix $\mathbf{A}$ is zero. This determinant will define a frequency function as follows:

$$f(\omega) = |\mathbf{A}| = 0. \tag{20}$$

The roots of Eq. (20) are called natural frequencies and can be determined numerically. In addition, the normalized mode shapes of the free vibrations of the beam can be determined by solving the three equations of Eq. (19) for c_2, c_3 and c_4 in terms of c_1 along with using one of the conventional normalization methods. Note that for constant values of α and β, Eq. (15) reduces to the analytical exact solution for uniform beams.

3.2. Homogeneous variable-section beam

Consider a variable-cross-section beam (Fig. 1) made of a homogeneous material with Young's modules of elasticity E and mass density ρ for which the section area function is s and the gyration radius r defines the second area moment of inertia of the section about the so-called neutral axis, as follows:

$$I = sr^2. \tag{21}$$

Setting the beam characteristics, the ϕ_i functions for a third-order WKB approximation of the transverse free vibration mode shapes of the beam can be simplified as

$$\phi_0 = \frac{1}{\kappa} \int_0^x \frac{1}{\sqrt{r}}\, \mathrm{d}\xi,$$

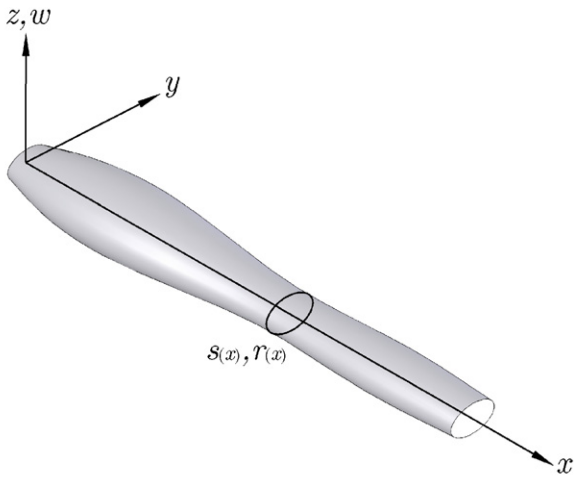

Fig. 1. Geometry of a homogeneous variable section beam.

$$\phi_1 = -\frac{1}{4}\ln rs^2,$$
$$\phi_2 = \frac{\kappa}{32}\int_0^x \mu\,\mathrm{d}\xi,$$
$$\phi_3 = -\frac{\kappa^2}{64}\mu\sqrt{r}, \tag{22}$$

where

$$\mu = 16\left(\frac{s''\sqrt{r}}{s} + \frac{s'r'}{s\sqrt{r}}\right) + 12\left(\frac{r''}{\sqrt{r}} - \frac{s'^2\sqrt{r}}{s^2}\right) + \frac{r'^2}{r\sqrt{r}} + \frac{8p}{Esr\sqrt{r}},$$
$$\kappa = \sqrt[4]{\frac{E}{\rho}}, \tag{23}$$

4. Examples and discussion

4.1. Linearly tapered cantilever beam

To examine the validity of the developed asymptotic approach we consider the problem of determining the natural frequencies and mode shapes of a linearly tapered cantilever beam as shown in Fig. 2 which displays a linear variation of height while the beam width is constant. Let us define the following non-dimensional frequency and length:

$$\xi = x/L, \quad \lambda = \frac{\omega}{\omega_0}, \tag{24}$$

where ω_0 is given by

$$\omega_0 = \sqrt{\frac{Er_0^2}{\rho L^4}} \tag{25}$$

and r_0 is the radius of gyration at the root of the beam. For the configuration shown in Fig. 2 the variation of beam cross section area and radius of gyration can be evaluated as

$$s = s_0\eta, \quad r = r_0\eta, \tag{26}$$

where s_0 is the cross section area at the root of the beam and $\eta = 1 - c\xi$, in which c is the taper ratio constant.

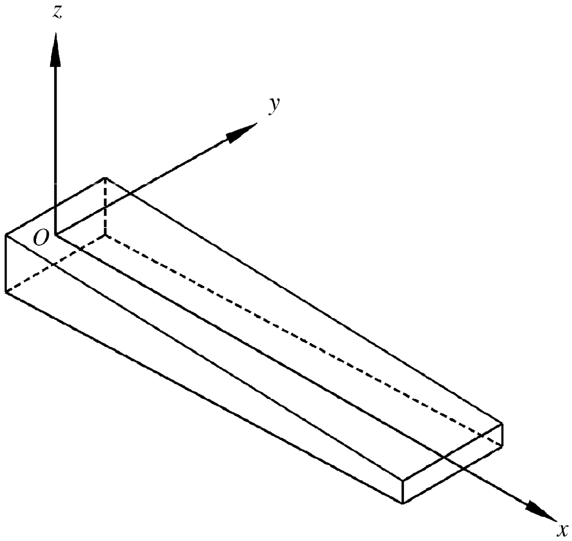

Fig. 2. Geometry of a tapered beam with linearly varying height.

The beam is clamped at its root (the origin point of xyz coordinate system) and is free at the tip. Thus the enforced boundary conditions at the clamped and free ends can be stated as following:

$$\bar{w}|_{x=0} = 0, \quad \bar{w}'\big|_{x=0} = 0, \quad \bar{w}''\big|_{x=L} = 0, \quad \bar{w}'''\big|_{x=L} = 0. \tag{27}$$

Using the first-order WKB solution and applying the boundary conditions, gives the transverse displacement of the beam as follows:

$$\bar{w} = c_1\eta^{-0.75}(\sin\psi - \sinh\psi + \gamma(\cos\psi - \cosh\psi)), \tag{28}$$

where

$$\begin{aligned} \psi &= 2c^{-1}\sqrt{\lambda}\left(\sqrt{\eta} - 1\right), \\ \gamma &= \left(\frac{a_1\sin\psi - a_2\sinh\psi + a_3(\cosh\psi - \cos\psi)}{a_1\cos\psi - a_2\cosh\psi + a_3(\sinh\psi + \sin\psi)}\right)\Bigg|_{\xi=1} \end{aligned} \tag{29}$$

and

$$\begin{aligned} a_1 &= \eta^{2.5}\left(21c^2 - 16\lambda\eta\right), \\ a_2 &= \eta^{2.5}\left(21c^2 + 16\lambda\eta\right), \\ a_3 &= 32c\eta^3\sqrt{\lambda}. \end{aligned} \tag{30}$$

Setting the determinant of the coefficient matrix equal to zero along with some simplifications yields the following equation for obtaining the beam natural frequencies:

$$\begin{aligned} f(\lambda) = \{&b_1(\sinh\psi\cos\psi - \cosh\psi\sin\psi) - b_2(\sinh\psi\cos\psi - \cosh\psi\sin\psi) \\ &+ b_3(1 + \cosh\psi\cos\psi) + b_4(1 - \cosh\psi\cos\psi)\}\big|_{\xi=1} = 0, \end{aligned} \tag{31}$$

where

$$\begin{aligned} b_1 &= 336c^3\sqrt{\lambda\eta}, \\ b_2 &= 512c(\lambda\eta)^{1.5}, \\ b_3 &= 256(\lambda\eta)^2, \\ b_4 &= 105c^4. \end{aligned} \tag{32}$$

Table 1
Non-dimensional second natural frequencies λ_i of the linearly height tapered beam vs. taper ratio c

c	λ_1			λ_2			λ_3		
	Ref. [13]	1st-order WKB	% Err.	Ref. [13]	1st-order WKB	% Err.	Ref. [13]	1st-order WKB	% Err.
0	3.516	3.516	0	22.039	22.039	0	61.73	61.73	0
0.1	3.559	3.559	0	21.338	21.339	0	58.98	58.973	0
0.2	3.608	3.609	0	20.621	20.618	0	56.192	56.188	0
0.3	3.667	3.668	0	19.881	19.87	0.1	53.322	53.313	0
0.4	3.737	3.738	0	19.114	19.091	0.1	50.354	50.326	0.1
0.5	3.824	3.822	0	18.317	18.279	0.2	47.265	47.216	0.1
0.6	3.934	3.919	0.4	17.488	17.428	0.3	44.025	43.953	0.2
0.7	4.082	4.056	0.6	16.625	16.534	0.5	40.588	40.461	0.3
0.8	4.292	4.236	1.3	15.743	15.599	0.9	36.885	36.719	0.4
0.9	4.631	4.515	2.5	14.931	14.679	1.7	32.833	32.528	0.9
0.99	5.214	5.118	1.9	14.967	14.47	3.3	29.727	28.942	2.6

Now using Eqs. (28) and (31), the first-order approximation of the non-trivial solutions of the characteristic equation can be numerically obtained which results in the natural frequencies and the mode shapes of the beam. By using the above formulation the first three non-dimensional natural frequencies of the beam are evaluated and shown in Table 1 in comparison with those given in Ref. [13] for a range of beam taper ratios varying from zero up to the limit value of 0.99. Fig. 3 illustrates the first three natural mode shapes for the case of $c = 0.5$ using the present solution along with those obtained by the dynamic stiffness method [13]. The comparison of the results shows a very good agreement between the two methods.

It is well known that the finite element and the other approximation methods become more and more unreliable at higher frequencies [13], while the proposed WKB approximation fulfills, in general, more accuracy for higher frequencies due to the decreasing of the perturbation parameter ε. The results given in Table 1 illustrate this matter as well. It is also concluded from Table 1 that the accuracy of the first approximation gradually decreases with the increase in taper ratio.

4.2. Rotating beam with linearly varying height and width

Since the second-order WKB approximation does not include the axial force distribution, higher order terms must be used to analyze free vibration of beams with axial force. For instance, free vibration of rotating beams is an interesting case of study in this field including axial force distribution. In this section a cantilever beam of linearly varying height and width as shown in Fig. 4 is used to examine the validity of the introduced approximation. Similar to the previous example, the cross section area and radius of gyration can be expressed as follows:

$$s = s_0\eta^2, \quad r = r_0\eta. \tag{33}$$

Note that the geometric properties given in Eq. (33) can be employed to describe a large number of linearly tapered beams with different cross sections, namely circular and rectangular cross sections, etc.

It is assumed that the tapered beam is rotating about the z-axis at constant angular velocity of Ω. Thus the axial force distribution over the beam is determined by

$$p(x) = \Omega^2 \int_x^L \rho s(\zeta)\zeta \, d\zeta. \tag{34}$$

Substituting Eqs. (33) and (34) into Eq. (22) and carrying out some algebraic simplifications results in the first four terms of the series in Eq. (16) as comes below:

$$\phi_0 = -2c^{-1}\sqrt{\lambda}\left(\sqrt{\eta} - 1\right),$$
$$\phi_1 = -\frac{5}{4}\ln\eta,$$

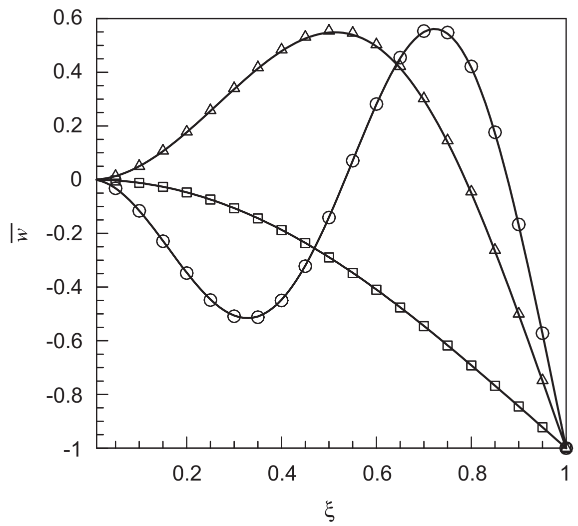

Fig. 3. First three natural mode shapes of the linearly height tapered beam with taper ratio of 0.5.

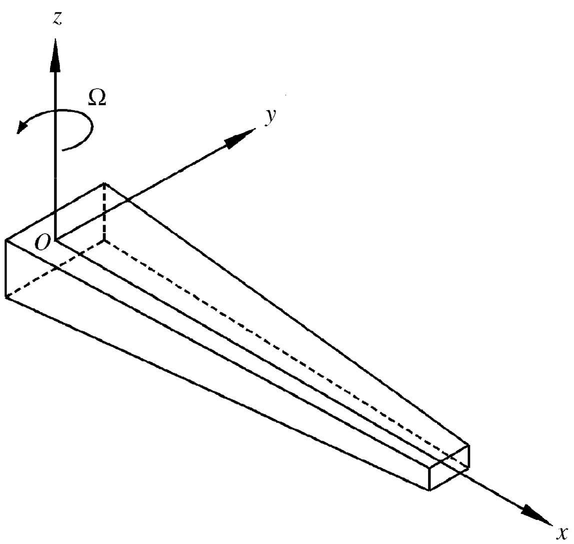

Fig. 4. Geometry of a tapered beam with linearly varying height and width.

$$\begin{aligned}\phi_2 = {} & \frac{17}{16}\frac{c}{\sqrt{\lambda\eta}} - \frac{1}{240}\frac{1}{c^3\sqrt{\lambda}}\left(\left(6c^4 - 16c^3 + 12c^2 - 32\right)\sigma^2 + 255c^4\right) \\ & + \frac{1}{120}\frac{\sigma^2}{c^3\eta^2\sqrt{\lambda\eta}}\left(5c^4\xi^4 - 30c^2\xi^2 + 40c\xi + \left(3c^4 - 8c^3 + 6c^2 - 16\right)\right), \\ \phi_3 = {} & -\frac{17}{64}\frac{c^2}{\lambda\eta} + \frac{1}{96}\frac{\sigma^2}{\lambda\eta^3}\left(3c^2\xi^4 - 8c\xi^3 + 6\xi^2 - \left(3c^2 - 8c + 6\right)\right), \end{aligned} \tag{35}$$

where σ is the non-dimensionalized rotational frequency of the beam as defined by

$$\sigma = \frac{\Omega}{\omega_0}. \tag{36}$$

The second- and third-order asymptotic approximations can be derived using Eq. (35). The boundary conditions of the considered rotating beam are the same conditions given in Eq. (27). The frequency equations for both second and third-order WKB approximations are obtained from setting the determinant of their corresponding coefficient matrices to zero. The frequency equations for these cases are long-winded and more complicated than that given in Eq. (31) and are not presented here. Since the coefficient matrix **A** is of small dimensions, its determinant can be simply calculated and thus the frequency equations can easily be solved numerically. For instance, a frequency marching procedure can be employed to find the natural frequencies and subsequently the corresponding mode shapes. The solution is carried out for the first three natural frequencies of a case as is shown in Fig. 4 using both second and third-order approximations. The taper ratio is taken to be 0.5 and the results are given in Tables 2–4 for a range of non-dimensionalized rotational frequencies in comparison with those given in Ref. [13]. Further, the natural mode shapes of the beam for the case with non-dimensionalized rotational frequency of 5.0 are shown in Fig. 5 for both third-order WKB asymptotic solution and method of Ref. [13]. Finally, Tables 2–4 and Fig. 5 demonstrate a good agreement between the proposed solution and Ref. [13] in the present case of study in which the effect of axial force distribution is included as well. Tables 2–4 also show a rapid convergence of the second-order approximation to the third-order approximation for calculating the natural frequencies chiefly in the first natural frequency. Moreover, from the results given in Tables 2–4 it can be concluded that the accuracy of the asymptotic solution increases for higher natural frequencies. Tables 2–4 also show that higher-order WKB approximations should be used to compute more accurate natural frequencies in higher rotational frequencies.

5. Conclusion

A WKB approximate based analytical solution of the free transverse vibrations of a class of variable-section beams is presented and used to obtain the corresponding natural frequencies and mode shapes. The third-order asymptotic solution are simplified for a homogeneous variable-cross-section beam and has been employed to study two sets of illustrative examples; First, a set of linearly tapered beams with various taper ratios are considered for determination of the natural frequencies and mode shapes applying a first-order

Table 2
Non-dimensional second natural frequency λ_1 of the linearly height and width tapered beam vs. rotational speed parameter σ

σ	Ref. [13]	1st-order WKB		2nd-order WKB		3rd-order WKB	
	λ_1	λ_1	Error %	λ_1	Error %	λ_1	Error %
0	4.625	4.531	2.1	4.573	1.2	4.642	0.3
1	4.764	—	—	4.642	3.1	4.822	1.2
2	5.156	—	—	4.822	13.7	5.279	2.4
3	5.746	—	—	5.279	25.7	5.896	2.7
4	6.473	—	—	5.896	36	6.644	2.7
5	7.29	—	—	6.644	43.7	7.538	3.4

Table 3
Non-dimensional second natural frequency λ_2 of the linearly height and width tapered beam vs. rotational speed parameter σ

σ	Ref. [13]	1st-order WKB		2nd-order WKB		3rd-order WKB	
	λ_2	λ_2	Error %	λ_2	Error %	λ_2	Error %
0	19.548	19.344	0.1	19.558	0.1	19.572	0.1
1	19.68	—	—	19.738	0.3	19.718	0.2
2	20.073	—	—	20.293	1.1	20.126	0.3
3	20.712	—	—	21.193	2.3	20.805	0.5
4	21.575	—	—	22.42	3.9	21.734	0.7
5	22.636	—	—	23.944	5.8	22.898	1.2

Table 4
Non-dimensional third natural frequency λ_3 of the linearly height and width tapered beam vs. rotational speed parameter σ

σ	Ref. [13]	1st-order WKB		2nd-order WKB		3rd-order WKB	
	λ_3	λ_3	Error %	λ_3	Error %	λ_3	Error %
0	48.5789	48.331	0.5	48.56	0.04	48.574	0.01
1	48.7073	—	—	48.754	0.1	48.747	0.08
2	49.0906	—	—	49.204	0.23	49.135	0.09
3	49.7227	—	—	49.945	0.45	49.779	0.11
4	50.5938	—	—	50.978	0.76	50.68	0.17
5	51.6918	—	—	52.28	1.14	51.823	0.25

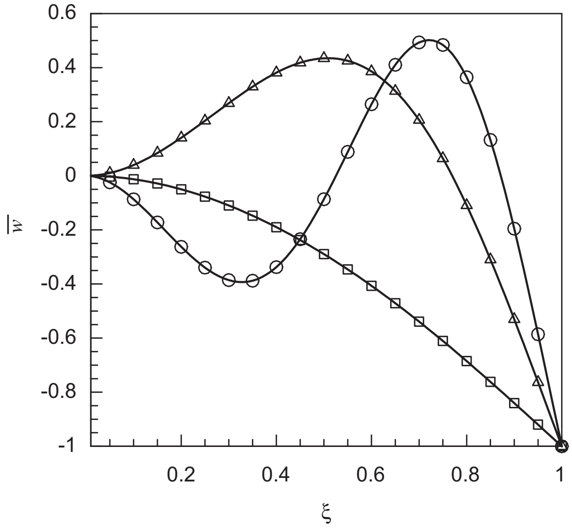

Fig. 5. First three natural mode shapes of the linearly height and width tapered beam with taper ratio of 0.5.

approximation. The comparison of the results with those in the literature demonstrates good agreement. The other examined test case is a rotating beam linearly tapered in both height and width. The additional term in this case with respect to the previous one is due to the inertial axial force distribution over the beam that affects the free vibration characteristics. The beam is studied for a range of non-dimensional rotational frequencies using second- and third-order approximate solutions in comparison with data available in the literature. It is concluded from the results that the asymptotic solution can be practically used for the beams including axial force distribution as well. As a significant conclusion, the obtained results show that in general the accuracy of the present WKB solution increases for higher frequencies while almost all of the other approximation approaches (e.g. finite element method, etc.) lose their reliability at higher frequencies. It is also observed from the results that the precision of the approximate solution decreases as the axial force magnitude or taper ratio increase and higher order terms are demanded to determine the free vibration characteristics more accurately.

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